Approximate efficiency in repeated games with correlated private signals*

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Abstract

This paper analyzes repeated games with hidden moves, in which players receive private signals and are able to communicate. We develop a model that obtains an efficiency result when private signals are correlated. A conditional probability approach is used to solve the learning problem that complicates players' incentives to cooperate in repeated private monitoring games with correlated signals and delayed communication. To avoid the learning problem, Compte (Econometrica 1998) has assumed private signals are independent, a condition that can be violated in some important applications such as the Bertrand oligopoly model.

Keywords: Bertrand oligopoly; private monitoring; delayed communication; effective independence.

1 Introduction

The theory of repeated games provides a formal structure to examine the possibility of cooperation in long-term relationships, such as collusion between firms,

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cooperation among workers, and international policy coordination. Earlier research in repeated games (e.g., Fudenberg and Maskin, 1986) has focused on games with perfect monitoring: games in which players can perfectly observe each other's past moves. It has been established that with perfect monitoring, efficient payoffs can be obtained in equilibrium under general conditions. However, perfect monitoring may not be available for a number of applications such as oligopoly models with uncertainty.

Relaxing the perfect monitoring assumption, subsequent research in repeated games develops in two directions. The literature on repeated games with imperfect public monitoring analyzes games where players observe a common public signal such as the dynamic Cournot oligopoly model of Green and Porter (1984). The literature on repeated games with imperfect private monitoring analyzes games where players each observe a separate private signal.

Repeated games with private monitoring admit a wide range of applications where only private monitoring is available. One example is the repeated Bertrand oligopoly model with uncertain demand. Currently there are two approaches to analyzing repeated games with imperfect private monitoring. The first approach attempts to determine whether the efficient outcome can be supported in equilibrium without assuming communication. Most papers (e.g., Sekiguchi, 1997; Piccione, 2002; Ely and Välimäki, 2002) in this category investigate equilibrium payoffs in the limit when the observation error converges to zero. The second approach assumes that communication is available to coordinate players' actions, for example, Compte (1998); Kandori and Matsushima (1998); Aoyagi (2002). This paper falls in the second category.

Compte (1998) has showed that the efficient payoffs can be approximated in a repeated game with strictly independent private signals and communication. In many real economic situations, however, it is inappropriate to assume strict independence. For example, in the Bertrand oligopoly model, there will typically exist macro or industry-wide shocks through which sales of competing firms are correlated. Based on this observation, we consider a symmetric n-player game with correlated private signals and show there exists an equilibrium in which players can get close to the efficient outcome payoffs even if the observation is far from zero.

The equilibrium strategy we propose is a a simple trigger strategy. As a first step we divide the repeated game into an infinite sequence of finitely repeated games, each of which consists of T standard stage games. We will refer to each series of T stage games as a *T-stage game*. In equilibrium players play a collusive action profile in all periods of the T-stage game and report private signals truthfully at the end of the T-stage game. Given the reported messages, players then decide whether to enter a punishment phase in which they play the static Nash equilibrium forever. With this equilibrium strategy, the efficient payoffs can be closely approximated even when private signals are correlated and the observation error is not close to zero. While the Compte (1998) model works only in games with three or more players, our model can be applied to games with two players as well as games with more than two players.

When private signals are correlated, a learning problem will occur in the T-stage game: players can use own observation to learn about signals received by their opponents during the T-stage game before private signals are reported publicly. As a result, " if a player learns that it is very unlikely he will be sanctioned, deterring deviations requires choosing a greater sanction compared to that where no such learning could have occurred."(Compte, 1998, page 598) Therefore Compte has assumed independent private signals to avoid this problem.

In contrast to Compte, we assume private signals are correlated and develop a solution to the learning problem. The solution we propose is rather straightforward. Given a player's private information, what she can learn in the T-stage game is the conditional probability of private signals received by her opponents. Learning does not pose a problem if this information is taken into account in determining whether to enter a punishment phase or not; players can weight the probability of entering a punishment phase by the inverse of the probability that a player's opponents observe those signals conditional on her own private information. With this adjustment, player i's expected probability of entering a punishment phase will be independent of any information she learns during the T-stage game. We shall call this approach *effective independence* to distinguish it from the strict independence assumed in Compte (1998). While both effective independence here and strict independence assumed in Compte ensure player i's incentive to cooperate is not affected by any private information received in the T-stage game, it is quite clear that effective independence does not exclude learning per se as strict independence does. In fact, with correlated signals, player i will be able to use her private information to learn about signals received by her opponents during the T-stage game.

Of course, for this conditional probability approach to work, one necessary condition is that players will report private signals truthfully in communication. This condition is satisfied when the probability of reverting to a punishment phase depends on the unanimity of reports made by all players at the end of the T-stage game. In this case players will have a strict incentive to report private signals truthfully if the distribution of private signals satisfies a correlation condition.

The main contribution of this work is to propose effective independence as a solution to the learning problem in repeated games with correlated private signals and communication, and to prove an efficiency result in a symmetric n-player game with correlated private signals. The efficiency result obtained here implies that a full cartel arrangement can be self-enforcing even if firms can make secret price cuts.

The remaining part of this paper is organized as follows: Section 2 introduces the concept of effective independence. Section 3 develops a symmetric n-player game with correlated private signals. Section 4 applies the effective independence approach to the n-player game and proves an efficiency result, while Section 5 applies the effective independence approach to a Prisoner's dilemma game and proves a folk theorem. Section 6 discusses alternative assumptions that may be used to obtain an efficiency result.

Table 1: Prisoner's dilemma game

	C	D
C	$(\pi,\ \pi)$	$(-L, \pi + d)$
D	$(\pi + d, -L)$	$(0, 0)$

2 Delayed communication and effective independence

DELAYED INFORMATION: Abreu et al. (1991) are the first to realize that information delay may enlarge the set of equilibrium payoffs in repeated games with imperfect monitoring. We illustrate their idea with a Prisoner's dilemma example.

Suppose two players, 1 and 2, play the Prisoner's dilemma game with expected payoff as shown in Table 1. Players do not observe their opponent's move, but they together observe a public signal $y \in \{\bar{y}, \underline{y}\}$, which is an imperfect indicator of moves made by both players. Suppose $\underline{y}$ occurs with probability τ if both players play C, while $\underline{y}$ occurs with probability μ ($\mu > \tau$) if one player deviates from C. Abreu et al. (1991, Proposition 3) has established that in this game, a player's maximum payoff v_i in a symmetric equilibrium equals the first-best value π minus the incentive cost $d/(\ell - 1)$ attributable to imperfect monitoring. That is,

$$v_i \leq \pi - \frac{d}{\ell - 1} \quad \text{if } \ell > 1 + \frac{d}{\pi}\,, \qquad \text{while } v_i = 0 \qquad \text{if } \ \ell \leq 1 + \frac{d}{\pi}. \tag{1}$$

The term ℓ ($\ell = \mu/\tau$) can be taken as a likelihood ratio reflecting how easily a deviation is detected.

Next we suppose that instead of observing a common signal every period, players only observe a sequence of public signals at the end of every T-periods. Abreu et al. (1991, Proposition 6) has showed that the delay of information release allows higher equilibrium payoffs; the maximum payoff now equals

$$v_i = \pi - \frac{1}{(1 + \delta \ldots + \delta^{T-1})} \frac{d}{(\ell - 1)},$$

which converges to π as $\delta \to 1$ and $T \to \infty$. Hence a T-period delay in revealing information reduces the incentive cost to $1/T$ of that under no delay. This is true because players' abilities to devise profitable cheating strategies would be diminished when information reporting is delayed or the number of periods of fixed action are increased.

PRIVATE MONITORING AND LEARNING: To the extent that the analysis of Abreu et al. (1991) can be applied, it can be used to obtain an approximate efficiency result in games with imperfect private monitoring. But there is one fundamental difference between games where private signals are reported publicly with a delay and games where public signals are observed with a lag. That is, in the T-stage game before private signals are reported publicly, a player may learn about information observed by her opponents based on her own private observation. We use a modified version of the above example to explain this difference. For simplicity, we assume that players' revelation constraints are satisfied; therefore, messages reported represent private signals truthfully. The *revelation constraints* refer to the conditions required to make players report private information truthfully. For illustration purpose, we set this complication aside, keeping in mind that this needs to be handled first in any actual construction of equilibrium strategies.

Suppose the private signal distributions are the same for both players. When both players play C, a player i, say $i = 2$, observes $\underline{y}$ with probability τ. When player 1 plays D while player 2 plays C, player 2 observes $\underline{y}$ with probability μ ($\mu > \tau$). Also suppose that the equilibrium strategy requires players to play CC in a T-stage game and to report private signals truthfully at the end of the T-stage game. Player 1 is punished if her opponent observes $\underline{y}$ at all dates in the T-stage game. Hence, *ex ante*, player 1 is punished with probability τ^T when both players conform to the equilibrium strategy.

At first we consider the case of independent private signals, i.e., $p(y_j|y_i, CC) = p(y_j|CC)$. To deter player 1 from deviating the size of punishment Δ in terms of loss in continuation payoffs needs only to be large enough to deter her from deviating at $t = 1$ for one period and following the equilibrium strategy thereafter.[1] The minimum

[1]Details can be found in the proof of Theorem 1 in Section 4.

punishment $\underline{\Delta}$ to deter deviation equals $(1-\delta)d/[\delta^T\tau^T(\ell-1)]$, where ℓ ($\ell=\mu/\tau$) is the likelihood ratio. Therefore the expected payoff v_1 equals

$$v_1=(1-\delta^T)\pi+\delta^T v_1-\delta^T\tau^T\underline{\Delta}=\pi-\frac{1}{(1+\delta\ldots+\delta^{T-1})}\frac{d}{(\ell-1)}, \tag{2}$$

which converges to π as δ goes to one and T goes to infinity. By symmetry this is also true of v_2. Thus, when private signals are independent the efficient payoffs can be approximated in equilibrium.

Next we consider the case of correlated private signals. Let $\rho_1=p(y_j=\underline{y}|y_i=\underline{y},CC)$ denote the probability that player j observes $\underline{y}$ when i observing $\underline{y}$. Let $\rho_0=p(y_j=\underline{y}|y_i=\bar{y},CC)$ denote the probability that player j observes $\underline{y}$ when i observing $\bar{y}$. When private signals are positively correlated but not perfectly correlated, $0<\rho_0<\tau$, and $1>\rho_1>\tau$.

It is still true that $\underline{\Delta}$ is large enough to deter player 1 from taking a one-period deviation at $t=1$. However, $\underline{\Delta}$ is not sufficient to deter deviations at some other time. For example, if at $t=(T-1)$ player 1 has received $\bar{y}$ only, i.e., $y_{1t}=\bar{y}$ for $t\leq(T-1)$. At this point the probability that player 2 would observe T number of $\underline{y}$ equals ${\rho_0}^{T-1}\tau$, which is significantly smaller than the *ex ante* expected probability τ^T. Deterring deviations at this point requires the size of punishment to be at least $\tilde{\Delta}$, $\tilde{\Delta}=[\tau^{T-1}\delta^{T-1}/{\rho_0}^{T-1}]\underline{\Delta}$. Ex ante, player 1's equilibrium payoff v_1 is bounded by $\pi-(\delta^T\tau^T\tilde{\Delta})/(1-\delta^T)$; thus it can be expressed as

$$v_1\leq\pi-\frac{(\tau\delta/\rho_0)^{T-1}}{(1+\delta\ldots+\delta^{T-1})}\frac{d}{(\ell-1)}\quad \text{if } \ell>1+\frac{(\tau\delta/\rho_0)^{T-1}}{(1+\delta\ldots+\delta^{T-1})}\frac{d}{\pi},$$

and $v_1=0$ otherwise. As $\tau>\rho_0$, the term $(\tau\delta/\rho_0)^{T-1}$ will be increasing exponentially in T for δ sufficiently close to one; therefore the payoff v_1 does not converge to π as δ goes to one and T goes to infinity.

One thing seems clear from this example. Learning increases the incentive costs to deter deviations, as deviation may be more profitable at time close to the end of the T-stage game. As a result, it is more difficult to obtain an approximate efficiency result in games with correlated private signals.

Compte (1998) has avoided this problem with the assumption of independent private signals. However, the assumption of independence can be violated in a number of interesting applications.

EFFECTIVE INDEPENDENCE: We propose a solution to the learning problem in games with correlated private signals. The logic underlying this solution is simple. When signals are correlated, player 1 can update on the probability she will be punished based on own private information received in the T-stage game, which may give her a stronger incentive to deviate. However, if player 1's own messages are also used in determining whether she should be punished, the newly acquired information will be irrelevant to her continuation payoff and thus, will not affect her incentive to cooperate.

Since $\rho_0 < \tau < \rho_1$, player 1 knows she is less likely to be punished conditional on observing $\bar{y}$ and more likely to be punished conditional on observing $\underline{y}$. However, if player 1 is punished with probability $\alpha\tau^T/({\rho_1}^{T-k}{\rho_0}^k)$ when she has observed $(T-k)$ number of $\underline{y}$ and k number of $\bar{y}$ while player 2 has observed T number of $\underline{y}$,[2] this information updating is no longer a problem. In this case player 1 expects to be punished with probability $\alpha\tau^T$ irrespective of her private information. For example, if at $(T-1)$ player 1 has observed a long sequence of $\bar{y}$, the chance that player 2 observes only $\underline{y}$ would be ρ_0^{T-1}. But the expected probability of punishment remains $\alpha\tau^T$, as ${\rho_0}^{T-1}\tau \cdot [\alpha\tau^{T-1}/{\rho_0}^{T-1}] = \alpha\tau^T$. Therefore, we can apply the Abreu et al. (1991) analysis to this game even if private signals are correlated.

In general we can apply this conditional probability approach, which we shall refer to as effective independence, to any games with correlated private signals, provided players' revelation constraints are satisfied. This approach has achieved the same effect of ensuring a player's incentive to cooperate is unaffected by her private information received in the T-stage game as that obtained in Compte (1998) with the assumption of strict independence. However, it is quite clear that effective independence does not exclude learning per se as Compte does.

[2] Here α is an appropriate chosen constant to ensure $\alpha\tau^T/({\rho_1}^{T-k}{\rho_0}^k)$ is less than or equal to one.

3 The basic model

THE STAGE GAME: There is a set of players $I = \{1, \ldots, n\}$ in the game. Players are symmetric; they have the same action space, same private signal distribution, and symmetric payoff functions. Each period a player can choose an action a_i from a finite set A, and an action profile a is a profile of actions played by the n players, a $= (a_1, \ldots, a_n) \in A^n$. Players do not observe each other's moves, but they each observe a private signal $y_i \in Y$ that is an imperfect indicator of the actions played by opponents. The set Y has finite number of elements. For each possible action profiles a and for each $y_i \in Y$, y_i will be observed with positive probability, $p(y_i|\mathbf{a}) > 0$. Private signals are positively correlated but not perfectly correlated; for any pair of players i, j ($i \neq j$) and for $y \in Y$, $1 > p(y_j = y|y_i = y, \mathbf{a}) > p(y_j = y|\mathbf{a})$.

Let $\mathbf{y} = (y_1, \ldots, y_n)$ be a vector of private signals. Let a_{-i} denotes the action profile played by player i's opponents, and y_{-i} be the private signals received by her opponents,

$$a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \qquad \text{and } y_{-i} = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n).$$

Player i's realized payoff $u(a_i, y_i)$ is a function of her own action and private signal only, and is independent of her opponents' actions a_{-i} and their private signals y_{-i}. Of course, player i's payoff is related to a_{-i} through y_i, which is a stochastic function of (a_i, a_{-i}). The expected payoff for player i from the action profile a equals $g_i(\mathbf{a})$,

$$g_i(\mathbf{a}) = \sum_{y_i \in Y} p(y_i|\mathbf{a}) u(a_i, y_i).$$

We assume that the stage game has a symmetric Nash equilibrium $(a^B, \ldots, a^B)$ whose payoff is normalized to zero.

THE REPEATED GAME: The repeated game $G(\delta)$ is an infinitely repeated stage game with players' preference represented by the discounting criterion. A player's private history consists of her own previous action choice a_i and privately observed signal y_i. Let $\iota_i^t \in \Im_i^t$ be the private history of player i at the beginning of period t before choosing a_i. This implies that $\iota_i^1 = \{0\}$, $\iota_i^2 = \{a_{i1}, y_{i1}\}$, and $\iota_i^t = \iota_i^{t-1} \times \{a_{it-1}, y_{it-1}\}$.

COMMUNICATION: In addition, player i can report a public message $m_{it} \in M_i$ at the end of each stage game, where M_i is player i's message space and $Y \subseteq M_i$. Players report their messages simultaneously.

Player i's history consists of two parts: her private history and the public history. The public history is simply the observed public signal sequence $P_i^t = (m_1, \cdots, m_{t-1})$. By definition, $P_i^t = P_j^t$ for all $i, j \in I$ and for all t. Thus we can omit the subscript i. Let H_i^t denote player i's history at time t, i.e., $H_i^t \equiv ((\iota_{i1}, P_1), \cdots, (\iota_{it-1}, P_{t-1}))$. We use ι_{i0}, P_0 to denote the null private and public histories in which nothing has happened.

A strategy for a player in the repeated game is a sequence $\sigma_i = (\sigma_{i1}, \sigma_{i2}, \ldots)$, where each σ_{it} is a function mapping the player's history H_i^t to the action set A and mapping $\Im_i^{t+1} \times P_i^t$ to her message space M_i for all $t \geq 2$. At $t = 1$ the null history is mapped to her action set A, while the initial private information ι_{i1} is mapped to M_i.

Players are risk-neutral, maximize long-run expected payoffs and have the same discount factor δ. Following standard practice, we normalize player i's net payoff $v_i(\sigma)$ in the repeated game to the stage game payoff. Hence, given a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ the expected payoff v_i for player i in the repeated game can be expressed as follows

$$v_i(\sigma) = (1 - \delta)E[\sum_{t=1}^{\infty} \delta^{t-1} g_{it} | \sigma].$$

The equilibrium concept is a special class of Nash equilibrium called *perfect public equilibrium*. A perfect public equilibrium is a profile of public strategies that constitutes a NE in the continuation game at any date t and for any history H_i^t. A strategy σ_i is a *public* strategy if at any time t, player i's action a_{it} depends only on the public history P^t and not on her private information while her report m_{it} depends on the most recent private information ι_{it-1}.

Although communication can take place every period, in equilibrium, players are required to send informative messages only at the end of every T ($T \geq 1$) periods. That is, players report a sequence of private signals observed at date $t \in \{T, 2T, \ldots\}$, but send no messages at all other dates.[3] In this case the most recent private history

[3]In an oligopoly model, one can imagine each colluding firm records sales each week, and reports

for player i at any date t consists of the action choice a_{it} and private signals y_{it} since date $kT+1$ where $kT < t \leq (k+1)T$, i.e., since the last time any communication took place. When communication takes place only every T periods, we abuse notations by denoting the sequence of messages reported by player i at the end of T-stage game by m_i^T, $m_i^T = \{m_{i1}, \dots, m_{iT}\}$. We will also use $m^T = \times_i m_i^T$ as the sequence of public messages reported by all players.

4 Approximate efficiency

One equilibrium of the repeated game specified in Section 3 is for all players to play the stage game NE a^B at all dates t and for all history H_i^t. In this section we determine the conditions that imply a symmetric collusive action profile a*, where $g_i(\mathbf{a}^*) = g^* > 0$ for all i, can be sustained as an equilibrium outcome, and compute the maximal equilibrium payoff that can be achieved in the n-player game.

We consider a simple *trigger strategy* to sustain the collusive equilibrium. Players start the first T-stage game by playing the collusive action profile a* at each date in the T-stage game, and report privately received signals truthfully at the end of the T-stage game. They revert to static NE forever with probability $\Phi(m^T)$ determined by reported messages at the end of the T-stage game, but continue playing a* in the next T-stage game with probability $1 - \Phi(m^T)$. In what follows we shall refer to $\Phi(m^T)$ as the *probability of sanctions*.

The *probability of sanctions* $\Phi(m^T)$ is constructed such that it is independent of any learning occurred in the T-stage game. Let **1** be an $(n-1)$-dimension vector of ones and γ be an appropriately chosen constant. For a message profile m_t reported for period t, we define a statistic $\tilde{q}(m)$ as

$$\tilde{q}_i(m_t) = \begin{cases} \frac{\gamma}{p(y_{-i}=y\cdot\mathbf{1}|y_i=y,\mathbf{a}^*)} & \text{if for all } j \in I,\ m_{jt} = y \quad \text{for some } y \in Y, \\ 0 & otherwise, \end{cases} \tag{3}$$

the weekly sales at the end of every month.

The probability $p(y_{-i} = y \cdot \mathbf{1} | y_i = y, \mathbf{a}^*)$ is the conditional probability that player i's opponents observe the same signal y as her own when players all follow the equilibrium strategy. The assumption of symmetry implies that $\tilde{q}_i(m_t)$ is the same for all players regardless their reported messages; therefore we can omit the subscript i.

If we let the *per period probability of sanctions* to be $(1 - \tilde{q}(m_t))$, then the probability of sanctions $\Phi(m^T)$ is a product of the per period probability of sanctions for all periods,[4]

$$\Phi(m^T) \equiv \prod_{t=1}^{T} [1 - \tilde{q}(m_t)]. \tag{4}$$

Clearly the probability of sanctions depends on the unanimity of messages reported by all players; an increase in the unanimity of messages increases the expected value of $\tilde{q}$ and reduces the expected value of Φ, vice versa. Because private signals are independent across periods the expected value of $\Phi(m^T)$ is just the product of the expected values of per period probability of sanctions. In particular, when players all play cooperatively in the T-stage game and report truthfully, the expected value of the statistic $\tilde{q}$ equals γ while the expected value of Φ equals $(1 - \gamma)^T$.

For the trigger strategy outlined above to be an equilibrium in the repeated game, we need two conditions. First, any deviation by a player i is statistically distinguishable. Second, player i has an incentive to report private information truthfully in communication. To ensure these two conditions, we make the following two assumptions.

Assumption 1. ($\textit{Distinguishability condition}$) *For all* $a_i \in A \backslash \{a_i^*\}$,

$$1 > \theta(a_i) = \sum_{y \in Y} \frac{p(y_{-i} = y \cdot \mathbf{1}, y_i = y | a_i, a_{-i}^*)}{p(y_{-i} = y \cdot \mathbf{1} | y_i = y, \mathbf{a}^*)}. \tag{5}$$

We can take $\theta(a_i)$ as the weighted sum of probabilities that players all observe the same signal when player i deviates while her opponents follow the equilibrium strategy, where the weights are the probabilities of player i's opponents observing the

[4]Again the assumption of symmetry ensures that the probability of sanctions $\Phi(m^T)$ is the same for all players.

same signal as her own when players all follow the equilibrium strategy. If we abuse the notation a bit, then when players all follow the equilibrium strategy, $a_i = a_i^*$,

$$\theta(a_i^*) = \sum_{y \in Y} p(y_i = y|\mathbf{a}^*) = 1.$$

The expected per period probability of sanction equals $(1-\gamma)$ when players all follow the equilibrium strategy, however, any unilateral deviation increases the expected value to $(1-\gamma\theta(a_i))$. If we define θ' as

$$\theta' = \max_{a_i \in A \setminus \{a_i^*\}} \theta(a_i),$$

then Assumption 1 implies that any deviation increases the per period probability of sanctions by at least $\gamma(1-\theta')$, provided private information is reported truthfully.

A sufficient condition for any deviations to be statistically distinguishable is that any unilateral deviation decreases the chance that players all observe a same signal. That is, for all $y \in Y$,

$$p(y_{-i} = y \cdot \mathbf{1}, y_i = y|a_i, a_{-i}^*) \le p(y_{-i} = y \cdot \mathbf{1}, y_i = y|\mathbf{a}^*),$$

and for at least one y,

$$p(y_{-i} = y \cdot \mathbf{1}, y_i = y|a_i, a_{-i}^*) < p(y_{-i} = y \cdot \mathbf{1}, y_i = y|\mathbf{a}^*).$$

Of course, a weaker condition implies the distinguishability condition.

The next condition imposes restrictions on the correlation of private signals.

Assumption 2. (*Correlation condition*) *The private signals are sufficiently correlated such that there exists* $\kappa > 1$*, for all* $a_i \in A$ *and for all* $y \in Y$*,*

$$\begin{aligned} &p(y_{-i} = y \cdot \mathbf{1}|y_i = y, \mathbf{a}^*) \ge \tfrac{1}{\kappa}, \\ &p(y_{-i} = y \cdot \mathbf{1}|y_i = y, a_i, a_{-i}^*) \ge \kappa \cdot \max_{y' \in Y \setminus y} p(y_{-i} = y' \cdot \mathbf{1}|y_i = y, a_i, a_{-i}^*) \, . \end{aligned} \tag{6}$$

Condition (6) implies that, for any action $a_i \in A$, and for any pair of signals $y, y' \in Y$ ($y' \neq y$),

$$\frac{p(y_{-i} = y \cdot \mathbf{1}|y_i = y, a_i, a_{-i}^*)}{p(y_{-i} = y' \cdot \mathbf{1}|y_i = y, a_i, a_{-i}^*)} \ge \frac{p(y_{-i} = y \cdot \mathbf{1}|y_i = y, \mathbf{a}^*)}{p(y_{-i} = y' \cdot \mathbf{1}|y_i = y', \mathbf{a}^*)}. \tag{7}$$

Since the weights used in constructing $\tilde{q}$ depend on the message profiles reported, player i will report strategically to increase (decrease) the value of $\tilde{q}$ (the probability of sanctions). However, if the joint distribution of signals satisfies the correlation condition, player i has no incentive to report a different signal y' while observing y. Misrepresenting lowers the chance of unanimous reports from $p(y_{-i} = y \cdot \mathbf{1} | y_i = y, a_i, a^*_{-i})$ down to $p(y_{-i} = y' \cdot \mathbf{1} | y_i = y, a_i, a^*_{-i})$. Hence player i can not gain from lying about private signals, for the expected value of $\tilde{q}$ from lying is actually lower than that from reporting truthfully,

$$p(y_{-i} = y' \cdot \mathbf{1} | y_i = y, a_i, a^*_{-i})\tilde{q}(y', \ldots, y') \leq p(y_{-i} = y \cdot \mathbf{1} | y_i = y, a_i, a^*_{-i})\tilde{q}(y, \ldots, y),$$

even if $\tilde{q}(y', \ldots, y')$ may be greater than $\tilde{q}(y, \ldots, y)$.

The correlation condition imposes restrictions on the relative magnitude of conditional probabilities, but not on the absolute magnitude of any conditional probabilities. In games where there are large number of private signals, the conditional probability that player i's opponents observe the same signal as her own can be far from one while still satisfying Assumption 2. This is different from the high correlation assumed in some previous work. For example, the "almost public monitoring" assumed in Mailath and Morris (2002) requires the conditional probability of a player's opponents all observing the same signal as her own to be sufficiently close to one.

The distinguishability condition and the correlation condition are sufficient conditions for our efficiency result; the former ensures players' incentives to cooperate while the latter ensures players' incentives to report truthfully. When Assumption 2 is satisfied, communicating the true observation is a best response for player i, irrespective of whether she conforms to the collusive action profile or not. This remains true even if she is required to report a sequence of signals.

Lemma 1. *Under Assumption 2, no player has an incentive to misrepresent private signals in communication.*

The construction of Φ ensures a player's expected probability of sanctions is independent of any learning occurred in the T-stage game.

Lemma 2. *In equilibrium, player i's expected probability of sanction in the T-stage game, at any time and for any private signals $(y_{i1}, \ldots, y_{it-1})$, remains the same as the ex ante expected probability of sanctions.*

Lemma 2 implies no learning in the T-stage game would affect players' incentive to cooperate. Thus an efficiency result can be obtained following Abreu et al. (1991).

Theorem 1. *Under Assumption 1 and Assumption 2, for all $\epsilon > 0$, there exists $\underline{\delta} \in (0,1)$ such that for all $\delta \geq \underline{\delta}$, there exists a collusive equilibrium $(\delta, T, \mathrm{a}^*)$ in which a player's payoff v^* satisfies*

$$v^* \geq g^* - \epsilon.$$

Before presenting the proof, we give an intuition to this result. Players revert to the static NE forever when they have reported different messages for all t in the T-stage game, however, they revert to NE with probability less than one if there are at least some periods for which they report the same messages. In this case, reporting truthfully is a best response for players, as misrepresenting private signals reduces the probability of unanimous reports and increases the probability of sanctions. Furthermore, players also have an incentive to play cooperatively in the T-stage game, as deviations increase the probability of sanctions.

Proof of Theorem 1. Denoting $\tilde{\Phi}$ as the expected probability of sanctions in equilibrium, we express a player's equilibrium payoff v^* as

$$v^* = (1 - \delta^T)g^* + \delta^T v^* - \delta^T \tilde{\Phi})v^*. \tag{8}$$

At first we show that players have no incentives to deviate for one period and follow the equilibrium strategy thereafter, provided they are sufficiently patient. The maximum gain player i can get from any one-period deviation in the T-stage game is bounded by $(1-\delta)(\bar{g} - g^*)$, where

$$\bar{g} = \max_{a_i' \in A} g_i(a_i', a_{-i}^*).$$

Meanwhile, a one period deviation increases the probability of sanctions by $[(1-\gamma\theta') - (1-\gamma)](1-\gamma)^{T-1}$. Player i has no incentive to take a one period deviation when the following condition is satisfied

$$(1-\delta)(\bar{g}-g^*) \leq \delta^T[(1-\gamma\theta')-(1-\gamma)](1-\gamma)^{T-1}v^* = \delta^T(\ell-1)\tilde{\Phi}v^* \ . \tag{9}$$

Here the likelihood ratio ℓ equals $(1-\gamma\theta')/(1-\gamma)$, which is strictly greater than one. We refer to condition (9) as player i's *incentive constraint* of no one-shot deviation. As ℓ is greater than one, the right-hand side term $\delta^T(\ell-1)\tilde{\Phi}v^*$ is strictly greater than zero, but the left-hand side term $(1-\delta)(\bar{g}-g^*)$ goes to zero as $\delta \to 1$. Hence there exists a $\underline{\delta}$ such that for all $\delta \geq \underline{\delta}$, player i has no incentive to take a one-shot deviation.

Next we show players have no incentives to deviate at all in the T-stage game when their incentive constraint of no one-shot deviation is satisfied. Consider player i's incentive to deviate for k ($k \leq T$) period. Deviating for k periods increases her payoff in the T-stage game by $(1-\delta^k)(\bar{g}-g^*)$, but also increases the probability of sanctions by $[(1-\gamma\theta')^k(1-\gamma)^{T-k}-(1-\gamma)^T]$. Thus player i has no incentives to deviate for k periods if

$$(1-\delta^k)(\bar{g}-g^*) \leq \delta^T[(1-\gamma\theta')^k(1-\gamma)^{T-k}-(1-\gamma)^T]v^* = \delta^T(\ell^k-1)\tilde{\Phi}v^*. \tag{10}$$

As $\delta < 1$, it is true that

$$(1-\delta^k)(\bar{g}-g^*) = (1-\delta)(1+\delta+\ldots\delta^{k-1})(\bar{g}-g^*) < k(1-\delta)(\bar{g}-g^*).$$

So we can simply her incentive constraint as follows

$$k(1-\delta)(\bar{g}-g^*) \leq \delta^T(\ell^k-1)\tilde{\Phi}v^*. \tag{11}$$

Because $(\ell^k-1) = (\ell-1)(1+\ell+\ldots\ell^{k-1}) > k(\ell-1)$, we conclude the inequality in (11) holds strictly if (9) is true. Therefore, player i has no incentives to deviate at all if it is not profitable for her to deviate for one-period in the T-stage game.

At last we compute players' payoff in the collusive equilibrium. We first note that for any $\delta \geq \underline{\delta}$, an appropriate choice of γ will ensure the incentive constraint (9) is exactly satisfied. In this case we can reformulate condition (9) as

$$\delta^T\tilde{\Phi}v^* = \frac{(1-\delta)(\bar{g}-g^*)}{(\ell-1)}.$$

Then we plug this equality into player i's equilibrium payoff function (8) and rearrange terms to get

$$v^* = g^* - \frac{1}{1+\delta\ldots+\delta^{T-1}}\frac{(\bar{g}-g^*)}{(\ell-1)}\ . \tag{12}$$

Hence we conclude that, for all $\epsilon > 0$, there exists (T,δ) such that $v^* \geq g^* - \epsilon$. $\square$

5 Folk theorem in Prisoner's dilemma game

In Section 4 we use a symmetric punishment (trigger strategy) to obtain an efficiency result in the n-player game with correlated private signals. By symmetric punishment we mean punishment scheme in which all players are punished simultaneously as in a trigger strategy. There, sustaining cooperation requires a distinguishability condition and correlation condition. In this section we show that weaker conditions may be enough to sustain cooperation when asymmetric punishment may be used. By asymmetric punishment we mean punishment scheme in which players are punished or rewarded differently according to the messages reported. An example where asymmetric punishment is used is the Fudenberg et al. (1994) model in which continuation payoff of a player suspected of deviating is transfered to her opponents.

As an example we consider a repeated Prisoner's dilemma game with expected payoffs as shown in Table 1. Suppose two players have the same signal space Y, $Y = \{\bar{y}, \underline{y}\}$. The marginal distribution of private signals is such that $p(y_i = \bar{y}|\mathbf{a}) = 1-\varepsilon$ when $\mathbf{a} = CC$, $p(y_i = \bar{y}|a_1a_2) = 1-\nu$ when $\mathbf{a} = CD$ or DC, and $p(y_i = \bar{y}|\mathbf{a}) = 1-\eta$ when $\mathbf{a} = DD$, where $\varepsilon < \nu < \eta$. For all action profiles, $p(y_i = \underline{y}|\mathbf{a}) = 1 - p(y_i = \bar{y}|\mathbf{a})$. The private signals of two players are correlated with correlation coefficient ρ and have joint distributions as shown in Table 2. Panel (a) is the joint distribution conditional on action profile CC, panel (b) is the joint distribution conditional on action profile CD or DC, and panel (c) is the joint distribution conditional on action profile DD. A similar distribution has been used by Bhaskar and van Damme (2002) in a Prisoner's dilemma game with private monitoring.

We define $\bar{\theta}$ as the probability of two players observing same signals when they

Table 2: Joint distribution of private signals

(a)

	$\bar{y}$	$\underline{y}$
$\bar{y}$	$(1-\varepsilon)^2 + \rho\varepsilon(1-\varepsilon)$	$(1-\rho)\varepsilon(1-\varepsilon)$
$\underline{y}$	$(1-\rho)\varepsilon(1-\varepsilon)$	$\varepsilon^2 + \rho\varepsilon(1-\varepsilon)$

(b)

	$\bar{y}$	$\underline{y}$
$\bar{y}$	$(1-\nu)^2 + \rho\nu(1-\nu)$	$(1-\rho)\nu(1-\nu)$
$\underline{y}$	$(1-\rho)\nu(1-\nu)$	$\nu^2 + \rho\nu(1-\nu)$

(c)

	$\bar{y}$	$\underline{y}$
$\bar{y}$	$(1-\eta)^2 + \rho\eta(1-\eta)$	$(1-\rho)\eta(1-\eta)$
$\underline{y}$	$(1-\rho)\eta(1-\eta)$	$\eta^2 + \rho\eta(1-\eta)$

both play C, $\hat{\theta}$ as the probability of observing same signals when one player plays C while the other plays D, and $\underline{\theta}$ as the probability of observing same signals when both play D. That is,

$$\bar{\theta} = p(y_1 = y_2|CC) = (1-\varepsilon)^2 + \varepsilon^2 + 2\rho\varepsilon(1-\varepsilon),$$

$$\hat{\theta} = p(y_1 = y_2|\mathbf{a} \in \{CD, DC\}) = (1-\nu)^2 + \nu^2 + 2\rho\nu(1-\nu),$$

$$\underline{\theta} = p(y_1 = y_2|DD) = (1-\eta)^2 + \eta^2 + 2\rho\eta(1-\eta).$$

To obtain a folk theorem in this game we need the following two conditions.

Assumption 3. *The joint signal distribution satisfies the condition* $\bar{\theta} > \hat{\theta} > \underline{\theta}$.

Assumption 4. *The private signals are correlated such that*

$$\min\left\{\frac{\nu-\varepsilon}{\nu}, \frac{\eta-\nu}{\eta}\right\} \leq \rho < 1.$$

Assumption 4 ensures the correlation condition in (7) of Section 4 is satisfied for the action profile a$^* = CC$, but only ensures a similar condition is satisfied for the player assigned to play C when DC or CD is to be played. Assumption 3 is weaker than the distinguishability condition in Section 4, which, however, will be satisfied for a$^* = CC$ in the Prisoner's dilemma game if the following conditions hold [5]

$$0 < \varepsilon < \nu < \frac{1}{2}, \quad \text{and}$$
$$1 > \rho > \frac{\nu(1-\nu) - \varepsilon(1-\varepsilon) - (\nu - \varepsilon)(1-2\nu)}{\nu(1-\nu) - \varepsilon(1-\varepsilon)}. \tag{13}$$

We show these relations formally as the following two facts.

Fact 1. *Suppose Assumption 4 is satisfied. For $a_i \in \{C, D\}$, $y \in \{\bar{y}, \underline{y}\}$ and $y' \neq y$,*

$$\frac{p(y_j = y|y_i = y, a_i, C)}{p(y_j = y'|y_i = y, a_i, C)} \geq \frac{p(y_j = y|y_i = y, CC)}{p(y_j = y'|y_i = y', CC)}. \tag{14}$$

For $a_1 \in \{C, D\}$, $y \in \{\bar{y}, \underline{y}\}$ and $y' \neq y$,

$$\frac{p(y_2 = y|y_1 = y, a_1, D)}{p(y_2 = y'|y_1 = y, a_1, D)} \geq \frac{p(y_2 = y|y_1 = y, CD)}{p(y_2 = y'|y_1 = y', CD)}. \tag{15}$$

For $a_2 \in \{C, D\}$, $y \in \{\bar{y}, \underline{y}\}$ and $y' \neq y$,

$$\frac{p(y_1 = y|y_2 = y, D, a_2)}{p(y_1 = y'|y_2 = y, D, a_2)} \geq \frac{p(y_1 = y|y_2 = y, DC)}{p(y_1 = y'|y_2 = y', DC)}. \tag{16}$$

Fact 2. *If (13) is satisfied, for $i = 1, 2$ and $j \neq i$,*

$$1 > \theta(D) = \sum_{y \in \{\bar{y}, \underline{y}\}} \frac{p(y_j = y_i = y|a_i = D, a_j = C)}{p(y_j = y|y_i = y, CC)}. \tag{17}$$

Hence it follows from Theorem 1 that, under condition (13) and Assumption 4, (π, π) can be approximated in equilibrium. Of course, weaker conditions are enough to sustain cooperation when asymmetric punishment is used; under Assumption 3 and 4, a folk theorem can be obtained in the Prisoner's dilemma game.

Theorem 2. *Under Assumption 3 and 4, any feasible, individually rational payoffs can be approximately obtained in perfect public equilibria as $T \to \infty$, provided the discount factor is sufficiently close to one.*

Like Compte (1998) and Kandori and Matsushima (1998) we will use the Fundenberg and Levine (1994) algorithm to prove Theorem 2.

[5]Note that $0 < \varepsilon < \nu < \frac{1}{2}$ implies $\nu(1-\nu) > \varepsilon(1-\varepsilon)$.

As a first step we transform the original game into an infinite sequence of T-stage games, where each stage of the transformed game $G^T(\beta)$ lasts for T periods. Let $\beta = \delta^T$ be the discount factor of the transformed game. At the beginning of each T-stage game players simultaneously select an action $f_i \in F_i$, where f_i consists of a sequence of action function $\{f_{it}\}_{t=1}^T$ and a message function m_i. The action function f_{it} maps player i's most recent history to the action set A while the message function m_i maps her most recent private history to her message space. Player i's T-stage game payoff $g_i^T(f)$ equals

$$g_i^T(f) = \frac{1-\delta}{1-\delta^T}\sum_{t=1}^{T}\delta^{t-1}g_i(\mathbf{a}_t|f).$$

Note that $g_i^T(f) = g_i(\mathbf{a})$ if a is played in all periods of the T-stage game.

Player i's payoff in the transformed game equals her T-stage game payoff $g_i^T(f)$ plus a side payment $S_i(m^T)$,

$$v_i = g_i^T(f) + E[S_i(m^T)|f]. \quad (18)$$

The side payment $S_i(m^T)$ can be taken as the variation in continuation payoffs w_i,

$$S_i(m^T) = \frac{\beta}{(1-\beta)}[w_i(m^T) - v_i].$$

In equilibrium players play a specified action profile $\mathbf{a}^*$ at all dates in the T-stage game. They make no reports until the end of the T-stage game, when they report a sequence of messages revealing private signals observed in the T-stage game. For all $\mathbf{a}^* \in A^2$ we define the set of equilibrium strategies for player i as $F_i^*(\mathbf{a}^*)$.

As the next step we introduce some notations and preliminary results. For a collusive action profile $\mathbf{a}^* \in \{CC, DC, CD\}$, we define a statistic q as follows[6]

$$q_i(m_t, \mathbf{a}^*) = \begin{cases} 0 & \text{if } m_{it} = m_{jt}, \\ \frac{\gamma(\mathbf{a}^*)}{p(y_j|y_i,\mathbf{a}^*)} & \text{if } \quad m_{it} = y_i \neq m_{jt} = y_j \text{ for } \ y_i, y_j \in \{\bar{y}, \underline{y}\}, \\ 1 & otherwise. \end{cases} \quad (19)$$

Here $\gamma(\mathbf{a}^*)$ is an appropriately chosen constant to ensure $q_i(m_t, \mathbf{a}^*)$ is less than or equal to one. The statistic q_i is different from the per period probability of sanctions

[6]Throughout this section we use j to denote player i's opponent.

$1-\tilde{q}(m_t)$ defined in Section 4; the former may in cases be different for the two players of the Prisoner's dilemma game, while the latter is always the same for all players of the n-player game.

Under an asymmetric punishment scheme, player i's incentive to cooperate may be based on rewards, in which case her side payment $S_i(m^T) \geq 0$, as well as on punishment, in which case her side payment $S_i(m^T) \leq 0$. When player i's incentive is based on punishment, we define

$$\Phi_i(m^T, \mathbf{a}^*) \equiv \prod_{t=1}^{T} q_i(m_t, \mathbf{a}^*).$$

In this case we may interpret q_i as the per period probability of sanctions and take $\gamma(\mathbf{a}^*)$ as the expected per period probability of sanctions in equilibrium.

When player i's incentive is based on rewards, we define

$$\Lambda_i(m^T, \mathbf{a}^*) \equiv \prod_{t=1}^{T} [1 - q_i(m_t, \mathbf{a}^*)].$$

In this case we may interpret $(1-q_i)$ as the per period probability of rewards and take $(1-\gamma(\mathbf{a}^*))$ as the expected per period probability of rewards in equilibrium.

One result that follows from this construction is that, under Assumption 4, players will have an incentive to report private signals truthfully.

Claim 1. *Let $y, y' \in \{\bar{y}, \underline{y}\}$ and $y \neq y'$. Under Assumption 4, for $i = 1, 2$ and for $a_i \in \{C, D\}$,*

$$\frac{p(y_j = y | y_i = y, a_i, C)}{p(y_j = y | y_i = y', CC)} \geq \frac{p(y_j = y' | y_i = y, a_i, C)}{p(y_j = y' | y_i = y, CC)}. \tag{20}$$

Under Assumption 4, for $a_1 \in \{C, D\}$,

$$\frac{p(y_2 = y | y_1 = y, a_1, D)}{p(y_2 = y | y_1 = y', CD)} \geq \frac{p(y_2 = y' | y_1 = y, a_1, D)}{p(y_2 = y' | y_1 = y, CD)}. \tag{21}$$

Under Assumption 4, for $a_2 \in \{C, D\}$,

$$\frac{p(y_1 = y | y_2 = y, a_2, D)}{p(y_1 = y | y_2 = y', DC)} \geq \frac{p(y_1 = y' | y_2 = y, a_2, D)}{p(y_1 = y' | y_2 = y, DC)}. \tag{22}$$

We first consider the case when $\mathbf{a}^* = CC$ is to be played as the collusive action profile. Given that player j plays cooperatively and report truthfully, player i needs to decide what to report while playing a_i, $a_i \in \{C, D\}$. Conditional on the action profile (a_i, C) and observing $y_i = \bar{y}$, the probability of j observing $\bar{y}$ equals $p(y_j = \bar{y}|y_i = \bar{y}, a_i, C)$, while the probability of j observing $\underline{y}$ equals $p(y_j = \underline{y}|y_i = \bar{y}, a_i, C)$. The expected per period probability of sanction equals $\gamma(CC)$ if player i reports $\bar{y}$, while it equals $\gamma(CC)p(y_j = \bar{y}|y_i = \bar{y}, a_i, C)/p(y_j = \bar{y}|y_i = \underline{y}, CC)$ ($> \gamma$) if she reports $\underline{y}$. She therefore has no incentive to misrepresent private signals. Similarly player i has no incentive to report $\bar{y}$ while observing $\underline{y}$.

We then consider the case when $\mathbf{a}^* = CD$ is to be played. In this case player 1 prefers to tell the truth, as lying increases (decreases) the probability of sanctions (rewards). In equilibrium player 2's continuation payoff will be independent of messages reported; she therefore has a weak incentive to report truthfully. By symmetry, when $\mathbf{a}^* = DC$ is to be played, both players will report truthfully.

Another result that follows from this construction is that, under Assumption 3, players will have an incentive to play cooperatively in equilibrium.

Claim 2. *Under Assumption 3, for any action profile* $\mathbf{a}^* \in \{CC, CD, DC\}$ *and for* $i = 1, 2$*, if player* i *is assigned to play C, playing D strictly increases the probability of nonunanimous messages such that*

$$\frac{p(y_i = \underline{y}, y_j = \bar{y}|D, a_j^*)}{p(y_j = \bar{y}|y_i = \underline{y}, \mathbf{a}^*)} + \frac{p(y_i = \bar{y}, y_j = \underline{y}|D, a_j^*)}{p(y_j = \underline{y}|y_i = \bar{y}, \mathbf{a}^*)} > 1. \tag{23}$$

We denote the left-hand side of (23) by ϕ_0 when $\mathbf{a}^* = CC$ ($a_j^* = C$), by ϕ_1 when $\mathbf{a}^* = CD$ ($a_j^* = D$), and by ϕ_2 when $\mathbf{a}^* = DC$ ($a_j^* = D$). When CC is to be played in the T-stage game, player i's deviation increases the expected probability of sanctions to $\gamma(CC)\phi_0$ or decreases the probability of rewards to $1 - \gamma(CC)\phi_0$; therefore she has no incentives to deviate. When DC or CD is to be played, the continuation payoff of the player assigned to play D will be independent of messages reported, while her opponent's continuation payoff depends on the unanimity of reports. In this case the incentive of the player assigned to play D is trivial; she has no reason to deviate from

D. When CD is to be played, player 1's deviation increases the expected probability of sanctions to $\gamma(CD)\phi_1$ or decreases the probability of rewards to $1-\gamma(CD)\phi_1$. When DC is to be played, player 2's deviation increases the expected probability of sanctions to $\gamma(DC)\phi_2$ or decreases the probability of rewards to $1-\gamma(DC)\phi_2$.

As a last step we apply the Fundenberg-Levine algorithm to compute the equilibrium payoff set. In doing that we solve the following optimization problem. For every welfare weight $\lambda \in \mathbb{R}^2$,

$$\max_{v,S} \lambda \cdot v \quad \textit{subject to}$$

(1) $v_i = g_i^T(f^*) + E[S_i(m^T)|f^*]$ for all i,

(2) $v_i \geq g_i^T(f_i', f_j^*) + E[S_i(m^T)|(f_i', f_j^*)]$ for all $f_i' \in F_i$ and for all i,

(3) $\lambda \cdot S(m^T) \leq 0$.

Denoting $k^*(\lambda, T)$ as the solution to the above linear programming problem, we define a *maximal half space* $H(T, \lambda)$ in the direction of λ as

$$H(T, \lambda) = \{v \in \mathbb{R}^2 | \lambda v \leq k^*\}.$$

Let Ω be the intersection of maximal half-spaces in direction of λ, $\Omega = \bigcap_{\lambda \in \mathbb{R}^2 \setminus \{0\}} H(T, \lambda)$, and denote the set of perfect public equilibrium payoffs by $E(\beta)$. It follows from Fudenberg and Levine (1994, Theorem 3.1) that a smooth compact convex subset of the interior of Ω is a subset of $E(\beta)$ for β close to 1.

Proof of Theorem 2. To establish Theorem 2 we identify points contained in the half-space $H(T, \lambda)$ for every direction $\lambda \neq 0$.

CASE 1: At first we consider the case where $\lambda_1, \lambda_2 > 0$ and the pure action profile CC maximizes λv. In this case players play CC in the T-stage game and report private signals truthfully at the end of T-stage game. For $i = 1, 2$, player i's side payment is determined as [7]

$$S_i(m^T) = -\frac{\Phi_i(m^T, \mathbf{a}^*)}{E[\Phi_i(m^T, \mathbf{a}^*)|f^*]} \frac{(1-\delta)d}{(1-\delta^T)(\phi_0 - 1)}. \quad (24)$$

[7] Claim 2 implies $\phi_0 > 1$.

First, player i has no incentive to misrepresent observed signals, as lying decreases the chance of unanimous reports and lowers her side payment as shown in (20). Second, player i has no incentives to deviate to D in the T-stage game. Consider a strategy f_i' consisting of playing D for one period at $t = 1$ and following the equilibrium strategy thereafter. Deviating for one period at $t = 1$ increases g_i^T by $(1-\delta)d/(1-\delta^T)$, but also decreases her side payment by

$$\left[\frac{(\gamma(CC)\phi_0)\gamma^{T-1}(CC)}{\gamma^T(CC)} - 1\right]\frac{(1-\delta)d}{(1-\delta^T)(\phi_0 - 1)} = \frac{(1-\delta)d}{(1-\delta^T)}.$$

Player i is therefore indifferent between deviating for one period at $t = 1$ and conforming to the collusive action, which also implies that she would have no incentive to deviate at all in the T-stage game.[8]

In equilibrium player i's expected side payment equals

$$E[S_i(m^T)|f^*] = -\frac{(1-\delta)}{(1-\delta^T)}\frac{d}{(\phi_0 - 1)},$$

which converges to zero as $\delta \to 1$ and $T \to \infty$. Hence the optimal value $k^*(\lambda, T)$ converges to $(\lambda_1 + \lambda_2)\pi$ as $\delta \to 1$ and $T \to \infty$.

CASE 2: Next we consider the case of $\lambda_1 > 0, \lambda_2 > 0$ but the pure action profile DC maximizes λv. In this case players play DC in the T-stage game and report truthfully at the end of the T-stage game. The side payment for player 1 is $S_1(m^T) = 0$ for all m^T, while the side payments for player 2 is [9]

$$S_2(m^T) = -\frac{\Phi_2(m^T, \mathbf{a}^*)}{E[\Phi_2(m^T, \mathbf{a}^*)|f^*]}\frac{(1-\delta)L}{(1-\delta^T)(\phi_2 - 1)}. \tag{25}$$

Player 1 is playing the static best response. She therefore has no incentive to deviate to C and has a weak incentive to report private signals truthfully. Player 2 has no incentive to misrepresent private signals, as this reduces her side payment. Moreover, she is indifferent between deviating to D at $t = 1$ for one period and not deviating; therefore she has no incentive to deviate at all. Since $E[S_2(m^T)|f^*]$ converges

[8] This can be proved in the same way as the proof of Theorem 1; if player i has no incentives to deviate for one period at $t = 1$, she would have no incentives to deviate for any k periods.

[9] By Claim 2, $\phi_2 > 1$.

to zero as $\delta \to 1$ and $T \to \infty$, the optimal value $k^*(\lambda, T)$ tends to $\lambda_1(\pi + d) - \lambda_2 L$ as $\delta \to 1$ and $T \to \infty$.

CASE 3: By symmetry, in the case where $\lambda_1 > 0, \lambda_2 > 0$ and pure action profile CD maximizes λv, $k^*(\lambda, T)$ tends to $\lambda_2(\pi + d) - \lambda_1 L$ as $\delta \to 1$, $T \to \infty$.

CASE 4: We then consider the case where $\lambda_1 \geq 0, \lambda_2 < 0$ and the action profile a $= DC$ maximizes λv. In this case players play DC in the T-stage game and report truthfully at the end of the T-stage game, but their incentives will be based on rewards. Let player 1's side payment be independent of messages reported, $S_1(m^T) = 0$ for all m^T. If we let

$$\varphi(\delta, T) = \sup_k \frac{(1-\delta^k)}{(1-\delta^T)} \frac{L}{1-\ell_2^k},$$

player 2's side payment equals

$$S_2(m^T) = \frac{\Lambda_2(m^T, \mathbf{a}^*)}{E[\Lambda_2(m^T, \mathbf{a}^*)|f^*]} \varphi(\delta, T). \tag{26}$$

In this case the likelihood ratio ℓ_2 equals $(1-\gamma(DC)\phi_2)/(1-\gamma(DC))$, which is less than 1.

Player 1 is playing the static best response and will have no incentive to deviate in the T-stage game. Player 2 has no incentive to misrepresent private signals, as this reduces her side payment. Deviating to D for k periods in the T-stage game increases g_i^T by $(1-\delta^k)L/(1-\delta^T)$, but also decreases her expected rewards by

$$\left[1 - \frac{(1-\gamma(DC))^{T-k}(1-\gamma(DC)\phi_2)^k}{(1-\gamma(DC))^T}\right] \varphi(\delta, T) = (1-\ell_2^k)\varphi(\delta, T).$$

Player 2 therefore has no incentives to deviate for any k period in the T-stage game.

In the appendix we show that $\varphi(\delta, T)$ converges to L and thus, the payoff v_2 converges to 0 as $\delta \to 1$ and $T \to \infty$. Hence the optimal value $k^*(\lambda, T)$ converges to $\lambda_1(\pi + d)$ as $\delta \to 1$ and $T \to \infty$.

CASE 5: The case where $\lambda_1 < 0, \lambda_2 \geq 0$ and the action profile CD maximizes λv is similar to the previous case. We can conclude that $k^*(\lambda, T)$ converges to $\lambda_2(\pi + d)$ as $\delta \to 1, T \to \infty$.

CASE 6: The case where $\lambda_1 < 0, \lambda_2 < 0$ and the action profile DD maximizes λv is trivial as DD is the stage game NE. In this case $k^*(\lambda, T) = 0$.

Hence we conclude that the intersection of half-spaces Ω contains a set arbitrarily close to the set of feasible individually rational payoffs as $\delta \to 1$ and $T \to \infty$. $\square$

6 Discussion

The *correlation condition* ensures player i's revelation constraint is satisfied regardless of her own play. In some applications this correlation condition can be violated when one player deviates from the collusive arrangement. For instance, if two firms play the Prisoner's dilemma game in which $\bar{y}$ stands for high sales and $\underline{y}$ stands for low sales, the correlation condition is violated as firm i's unilateral deviation to D makes high sales more likely for itself but less likely for its opponent. However, an approximate efficiency can still be obtained in this case.

Suppose private signals are correlated as defined in Section 5 conditional on CC being played. That is, $p(y_i = \underline{y}|CC) = \varepsilon$, $p(y_1 = y_2 = \bar{y}|CC) = (1-\varepsilon)^2 + \rho\varepsilon(1-\varepsilon)$ and $p(y_1 = y_2 = \underline{y}|CC) = \varepsilon^2 + \rho\varepsilon(1-\varepsilon)$. However, conditional on CD or DC being played, private signals are independent, $p(y_i|y_j, \text{a}) = p(y_i|\text{a})$ ($i \neq j$) for a $= CD$ or DC. Let $p(y_1 = \underline{y}|CD) = p(y_2 = \underline{y}|DC) = \xi$, and $p(y_2 = \underline{y}|CD) = p(y_1 = \underline{y}|DC) = \zeta$. Deviation makes high sales more likely for the deviating firm and less likely for its opponent, $\xi > \varepsilon > \zeta$.

Although the correlation condition in (6) is violated, we can get an efficiency result if the correlation coefficient ρ is sufficiently close to 1. In particular, if

$$\xi < \varepsilon + \rho(1-\varepsilon), \tag{27}$$

the efficient outcome payoff (π, π) can be approximated with the trigger strategy specified in Section 4.

To see why (27) implies an efficiency result we note that on equilibrium path, player i has a strict incentive to tell the truth, provided $\rho > 0$. However, when player i deviates to D she may have an incentive to misrepresent private signals. Conditional on ($a_i = D, a_j = C$) private signals are independent. If player i plays D and reports $\bar{y}$ the expected per period probability of sanctions equals $1 - \gamma(1-\xi)/[(1-\varepsilon) + \rho\varepsilon]$.

This is strictly greater than $1-\gamma$ as $(1-\xi) < (1-\varepsilon) + \rho\varepsilon$. When player i plays D and reports $\underline{y}$ the expected per period probability of sanctions equals $1-\gamma\xi/[\varepsilon+\rho(1-\varepsilon)]$, which is also greater than $1-\gamma$ given the condition in (27). Consequently deviating to D decreases the probability of unanimous reports and increases the probability of reverting to static NE; if players are patiently enough they have no incentives to deviate.

While making the alternative assumption does not affect the efficiency result, however, we will not be able to obtain the folk theorem. It is difficult to sustain any asymmetric payoffs in equilibrium, mainly because the revelation constraints are not satisfied when CD or DC is to be played in equilibrium.

The method used to induce truthful reporting in this model is closely related to a recent paper by Aoyagi (2002), which works with a repeated Bertrand game with correlated signals and communication. However, there are three differences between the two works. First, the current work deals only with a model of finite action space and signal space while Aoyagi (2002) has assumed continuous action space and signal distribution. Second, with delayed communication the condition required to sustain cooperation in our model is less restrictive than that in Aoyagi (2002). Although Aoyagi (2002) model assumes a continuous action space and signal distribution, sustaining collusion requires any price deviation to " have a discontinuous effect" on firms' sales. In particular, he requires any deviation from the collusive price to affect firms' sales to be so large that the likelihood ratio ℓ is greater than one plus the ratio of extra gain from deviation over the stage game payoff from collusion.[10] Third, the equilibrium payoff v_i in Aoyagi (2002) is uniformly bounded away from the efficiency frontier by the likelihood ratio ℓ as pointed out in Abreu et al. (1991).

However, given the information structure required to enforce cooperation, we are not sure whether the model developed here can be extended to allow for continuous

[10]Note that the collusive equilibrium payoff v_i in Aoyagi (2002) can be expressed in the same form as (1) in Section 2. This can be easily done by combining the incentive constraint equation and equilibrium payoff equation in Aoyagi model.

signal space. We do not believe a version of Assumption 1 and 2 can be satisfied in games with continuous signal spaces.

Appendix

Proof of Lemma 1. Given the strategy profile, player i has an incentive to increase the value of $\tilde{q}$ to reduce the chance of entering a punishment phase. Conditional on observing a private signal y at period t, player i's expected per period probability of sanctions is

$$E[(1-\tilde{q})|y_i = y, a_i, a^*_{-i}] = 1 - \gamma \frac{p(y_{-i} = y \cdot \mathbf{1}|y_i = y, a_i, a^*_{-i})}{p(y_{-i} = y \cdot \mathbf{1}|y_i = y, \mathbf{a}^*)} \tag{A.1}$$

if she reports truthfully. However, if she reported a different signal y', her expected per period probability of sanctions would be

$$1 - \gamma \frac{p(y_{-i} = y' \cdot \mathbf{1}|y_i = y, a_i, a^*_{-i})}{p(y_{-i} = y' \cdot \mathbf{1}|y_i = y', \mathbf{a}^*)}. \tag{A.2}$$

Under Assumption 2, for any $a_i \in A$ and for any pair of signals $y \neq y'$,

$$\frac{p(y_{-i} = y \cdot \mathbf{1}|y_i = y, a_i, a^*_{-i})}{p(y_{-i} = y' \cdot \mathbf{1}|y_i = y, a_i, a^*_{-i})} \geq \frac{p(y_{-i} = y \cdot \mathbf{1}|y_i = y, \mathbf{a}^*)}{p(y_{-i} = y' \cdot \mathbf{1}|y_i = y', \mathbf{a}^*)} \Longrightarrow$$
$$\frac{p(y_{-i} = y \cdot \mathbf{1}|y_i = y, a_i, a^*_{-i})}{p(y_{-i} = y \cdot \mathbf{1}|y_i = y, \mathbf{a}^*)} \geq \frac{p(y_{-i} = y' \cdot \mathbf{1}|y_i = y, a_i, a^*_{-i})}{p(y_{-i} = y' \cdot \mathbf{1}|y_i = y', \mathbf{a}^*)}.$$

Hence the expected probability of sanctions from truthful reporting is less than that from reporting a different signal y'; player i has no incentives to misrepresent private signals.

□

Proof of Lemma 2. Given player i's probability of sanctions, her *ex ante* expected probability of sanctions is

$$E[\Phi] = \prod_{s=1}^{T} \left[1 - \gamma \sum_{y \in Y} \frac{p(y_{-i} = y \cdot \mathbf{1}, y_i = y|\mathbf{a}^*)}{p(y_{-i} = y \cdot \mathbf{1}|y_i = y, \mathbf{a}^*)}\right] = (1-\gamma)^T. \tag{A.3}$$

Conditional on observing a sequence of signals $(y_{i1}, \ldots, y_{it-1})$, her expected probability of sanctions equals

$$E[\Phi_i|y_i^{t-1}] \;=\; E\left[\prod_{s=1}^{t-1}(1-\tilde{q}(\mathbf{y}))|y_i\right] E\left[\prod_{s=t}^{T}(1-\tilde{q}(\mathbf{y}))\right] = (1-\gamma)^T.$$

Thus, at any time t in the T-stage game and given any private signals observed, player i's expected probability of sanctions remains the same as her *ex ante* expected probability of sanctions. □

Proof of Fact 1. First, we show the condition (14) is satisfied at $\mathrm{a}^* = CC$.

(i) Suppose player j follows the equilibrium strategy, we check the condition for player i ($i \neq j$) when she plays C. When $a_i = C$, the condition (14) is satisfied if $p(y_j = y'|y_i = y, CC) \leq p(y_j = y'|y_i = y', CC)$, which holds true for $y = \bar{y}$ ($y' = \underline{y}$) and $y = \underline{y}$ ($y' = \bar{y}$). This is because

$$(1-\rho)\varepsilon < \varepsilon + \rho(1-\varepsilon), \quad (1-\rho)(1-\varepsilon) < (1-\varepsilon) + \rho\varepsilon.$$

(ii) Suppose player j follows the equilibrium strategy, we check the condition for player i ($i \neq j$) when she plays D. Conditional on $a_i = D$ and $y_i = \bar{y}$, the condition (14) becomes

$$\frac{(1-\nu)+\rho\nu}{(1-\rho)\nu} \geq \frac{(1-\varepsilon)+\rho\varepsilon}{\varepsilon+\rho(1-\varepsilon)}.$$

When $\rho \geq (\nu-\varepsilon)/\nu$, the left-hand side (LHS) is greater than $(1-\varepsilon)/\varepsilon$, while the right-hand side (RHS) is less than $(1-\varepsilon)/\varepsilon$. This follows from the fact that for two fractions $a/c > b/e$, $b/e < (a+b)/(c+e) < a/c$.

Conditional on $a_i = D$ and $y_i = \underline{y}$, the condition (20) becomes

$$\frac{\nu+\rho(1-\nu)}{(1-\rho)(1-\nu)} \geq \frac{\varepsilon+\rho(1-\varepsilon)}{(1-\varepsilon)+\rho\varepsilon}.$$

This holds strictly since $\nu > \varepsilon$ and $(1-\varepsilon) > (1-\nu)$.

Second, we show the condition (15) is satisfied.

(i) Suppose player 1 plays C. Conditional on $a_1 = C$, the condition (15) is satisfied if $p(y_2 = y'|y_1 = y, CD) \leq p(y_2 = y'|y_1 = y', CD)$, which is true for $y = \{\bar{y}, \underline{y}\}$.

(ii) Suppose player 1 plays D. Conditional on $a_1 = D$ and $y_1 = \bar{y}$, the condition (21) becomes

$$\frac{(1-\eta)+\rho\eta}{(1-\rho)\eta} \geq \frac{(1-\nu)+\rho\nu}{\nu+\rho(1-\nu)}.$$

When $\rho > (\eta-\nu)/\eta$, the left-hand side (LHS) is greater than $(1-\nu)/\nu$, while the right-hand side (RHS) is less than $(1-\nu)/\nu$. Conditional on $a_1 = D$ and $y_1 = \underline{y}$, the condition (21) becomes

$$\frac{\eta+\rho(1-\eta)}{(1-\rho)(1-\eta)} \geq \frac{\nu+\rho(1-\nu)}{(1-\nu)+\rho\nu},$$

which holds strictly when $\eta > \nu$.

The proof of (16) is similar to that of (15). □

Proof of Fact 2. To see why this is true, we note that (5) is satisfied at CC if it is true

$$\frac{(1-\nu)^2+\rho\nu(1-\nu)}{1-\varepsilon+\rho\varepsilon} + \frac{\nu^2+\rho\nu(1-\nu)}{\varepsilon+\rho(1-\varepsilon)} < 1.$$

This is equivalent to

$$\begin{aligned}[\varepsilon(1-\varepsilon)-\nu(1-\nu)]\rho^2 + [(1-\varepsilon)^2+\varepsilon^2-(1-\varepsilon)(1-\nu)^2-\nu(1-\nu)-\varepsilon\nu^2]\rho \\ +[\varepsilon(1-\varepsilon)-\varepsilon(1-\nu)^2-\nu^2(1-\varepsilon)] > 0.\end{aligned} \tag{A.4}$$

Since

$$(1-\varepsilon)^2+\varepsilon^2-(1-\varepsilon)(1-\nu)^2-\nu(1-\nu)-\varepsilon\nu^2 = \nu(1-\nu)+\varepsilon(1-\nu)^2+\nu^2(1-\varepsilon)-2\varepsilon(1-\varepsilon),$$

the condition (A.4) is equivalent to

$$\begin{aligned}[\varepsilon(1-\varepsilon)-\nu(1-\nu)]\rho^2 + [\nu(1-\nu)+\varepsilon(1-\nu)^2+\nu^2(1-\varepsilon)-2\varepsilon(1-\varepsilon)]\rho \\ +[\varepsilon(1-\varepsilon)-\varepsilon(1-\nu)^2-\nu^2(1-\varepsilon)] > 0.\end{aligned} \tag{A.5}$$

After some transformation, we get

$$(-\rho+1)\{[\nu(1-\nu)-\varepsilon(1-\varepsilon)]\rho - [\varepsilon(1-\nu)^2+\nu^2(1-\varepsilon)-\varepsilon(1-\varepsilon)]\}. \tag{A.6}$$

Obviously, (A.6) is strictly positive if and only if

$$1 > \rho > \frac{\varepsilon(1-\nu)^2+\nu^2(1-\varepsilon)-\varepsilon(1-\varepsilon)}{\nu(1-\nu)-\varepsilon(1-\varepsilon)}.$$

But $\varepsilon(1-\nu)^2+\nu^2(1-\varepsilon) = \nu(1-\nu)-(\nu-\varepsilon)(1-2\nu)$, which leads to (13). □

Proof of Claim 1. First, we show the condition (20) is satisfied at $\mathrm{a}^* = CC$.

(i) Suppose player j follows the equilibrium strategy, we check the condition for player i when $a_i = C$. The condition (20) is satisfied if $p(y_j = y|y_i = y', CC) \leq p(y_j =$

$y|y_i = y, CC)$. The condition holds strictly for the case of $y = \bar{y}$ ($y' = \underline{y}$) and the case of $y = \underline{y}$ ($y' = \bar{y}$) as

$$(1-\rho)\varepsilon < \varepsilon + \rho(1-\varepsilon), \quad (1-\rho)(1-\varepsilon) < (1-\varepsilon) + \rho\varepsilon.$$

(ii) We check the condition for player i when she plays D. Conditional on $a_i = D$ and $y_i = \bar{y}$, the condition (20) becomes

$$\frac{(1-\nu)+\rho\nu}{(1-\rho)(1-\varepsilon)} \geq \frac{(1-\rho)\nu}{(1-\rho)\varepsilon} = \frac{\nu}{\varepsilon},$$

which is satisfied when $\rho \geq (\nu - \varepsilon)/\nu$. Conditional on $a_i = D$ and $y_i = \underline{y}$, the condition (20) becomes

$$\frac{\nu + \rho(1-\nu)}{(1-\rho)\varepsilon} \geq \frac{(1-\rho)(1-\nu)}{(1-\rho)(1-\varepsilon)},$$

which holds strictly as $\nu > \varepsilon$ and $(1-\varepsilon) > (1-\nu)$.

Second, we show the condition (21) is satisfied.

(i) Suppose player 1 plays C. Conditional on $a_1 = C$, the condition (21) is satisfied if $p(y_2 = y|y_1 = y', CD) \leq p(y_2 = y|y_1 = y, CD)$, which is true for $y \in \{\bar{y}, \underline{y}\}$.

(ii) Suppose player 1 plays D. Conditional on $a_1 = D$ and $y_1 = \bar{y}$, the condition (21) becomes

$$\frac{(1-\eta)+\rho\eta}{(1-\rho)(1-\nu)} \geq \frac{(1-\rho)\eta}{(1-\rho)\nu},$$

which is satisfied when $\rho \geq (\eta - \nu)/\eta$. Conditional on $a_1 = D$ and $y_1 = \underline{y}$, the condition (21) becomes

$$\frac{\eta + \rho(1-\eta)}{(1-\rho)\nu} \geq \frac{(1-\rho)(1-\eta)}{(1-\rho)(1-\nu)},$$

which holds strictly when $\eta > \nu$.

The proof of (22) is similar to that of (21). □

Proof of Claim 2. First, we show $\phi_0 > 1$ for $i = 1, 2$. By symmetry we only need to show the condition holds for player 1 at CC. To do that, we plug the the values of

$p(y_2|y_1, CC)$ and $p(y_2|y_1, DC)$ into the formula, which gives

$$\begin{aligned}\phi_0 &= \frac{p(y_1 = \bar{y}, y_2 = \underline{y}|DC)}{p(y_2 = \underline{y}|y_1 = \bar{y}, CC)} + \frac{p(y_1 = \underline{y}, y_2 = \bar{y}|DC)}{p(y_2 = \bar{y}|y_1 = \underline{y}, CC)} \\ &= \frac{(1-\rho)\nu(1-\nu)}{(1-\rho)\varepsilon} + \frac{(1-\rho)\nu(1-\nu)}{(1-\rho)(1-\varepsilon)} \\ &= \frac{\nu(1-\nu)}{\varepsilon(1-\varepsilon)},\end{aligned}$$

which is greater than 1 by Assumption 3. Note Assumption 3 implies $2(1-\rho)\nu(1-\nu) > 2(1-\rho)\varepsilon(1-\varepsilon)$.

Next we show $\phi_1 > 1$. Plugging the values of $p(y_2|y_1, CD)$ and $p(y_2|y_1, DD)$ into the formula gives

$$\phi_1 = \frac{(1-\rho)\eta(1-\eta)}{(1-\rho)\nu} + \frac{(1-\rho)\eta(1-\eta)}{(1-\rho)(1-\nu)} = \frac{\eta(1-\eta)}{\nu(1-\nu)},$$

which is strictly greater than 1 by Assumption 3.

The proof of $\phi_2 > 1$ is similar to that of $\phi_1 > 1$. □

Lemma 3. *As $\delta \to 1$, $T \to \infty$, $\varphi(\delta, T)$ converges to L.*

Proof. Note that

$$\frac{(1-\delta^k)}{(1-\delta^T)} \frac{L}{(1-\ell_2^k)} = \frac{(1-\delta)L}{(1-\delta^T)(1-\ell_2)} \frac{(1+\delta+\ldots+\delta^{k-1})}{(1+\ell_2+\ldots+\ell_2^{k-1})},$$

and thus when δ is sufficiently close to one,

$$\varphi(\delta, T) = \sup_k \frac{(1-\delta^k)}{(1-\delta^T)} \frac{L}{(1-\ell_2^k)} = \frac{L}{1-\ell_2^T}.$$

As $\ell_2 < 1$, it is true that

$$\lim_{T\to\infty} \frac{L}{1-\ell_2^T} = L,$$

which implies that $\varphi(\delta, T)$ converges to L as $\delta \to 1$ and $T \to \infty$. □

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